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Calabi–Yau n-folds in projective superspace ☆

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Abstract

We show that Calabi–Yau n-folds serve as (partial) moduli spaces for a large class of supersymmetric field theory models, constructed in spacetimes of various dimensions and signatures, and with various numbers of supersymmetry generators.

0. Physics applications of supersymmetry have been explored and studied for well over four decades, much of which by using superspace techniques [1–3]. The rigorous mathematical aspects of superspace are also well studied [4,5]; see also [6] for curved spacetime and some general ("functorial") aspects. Calabi–Yau 3-folds (and later also 4-folds) have been introduced into physics as the target space for the "excess" dimensions in superstring theory and its M- and F-theory extensions [7–11]. In such physics applications, Calabi–Yau n-folds are algebraic varieties including certain of their simpler singularizations. It is then a little surprising that it has not been recognized so far that Calabi–Yau n-folds also appear amongst even the already known constructions of N-extended supersymmetry. To exhibit this, we will employ the formalism of projective superspace [12–18].

1. We consider the N-extended supersymmetry in flat spacetime ($\mathbb{R}^{t,s}$), for which the operators $\{D_i,\ \bar{D}^i,\ \partial\!\!\!/\}$ with $i = 1, \ldots, N$ provide a basis of left-derivatives in N-extended superspace [1,3,19] the elements of which satisfy the algebra:

☆ In honor of Prof. S.T. Yau's 65th birthday.

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$$\{D_i\,,\,D_j\} = 0 = \{\bar{D}^i\,,\,\bar{D}^j\}, \qquad \{D_i\,,\,\bar{D}^j\} = 2\mathrm{i}\,\delta_i^j\,\partial\!\!\!/, \qquad i,j = 1,\ldots,N. \tag{1}$$

The corresponding generators $Q_i, \bar{Q}^i$ of the N-extended supersymmetry themselves close the same algebra, but act as right-derivatives on the same superspace and so anticommute with the $D_i, \bar{D}^i$. The physics literature typically focuses one time-like ($t = 1$), and $s \in [0, 10]$ space-like dimensions, the upper limit set by string theory and its M-theory extension [9,10,20]. We omit the commutator relations which state that for each fixed $i = 1,\ldots,N$, $\mathrm{Span}(D_{\alpha\, i})$ and $\mathrm{Span}(\bar{D}^i_{\dot\alpha})$ transform as the ith copy of the two respective minimal spin-$\frac{1}{2}$ representations[1] $\mathscr{S}$ and $\bar{\mathscr{S}}$ of $\mathrm{Spin}(t,s)$. The indices $\alpha, \dot\alpha$ specify respective basis elements for $\mathscr{S}$ and $\bar{\mathscr{S}}$, and will be omitted herein for the most part as was done in (1). Using the Feynman slash notation as in Refs. [1,3], $\partial\!\!\!/_{\alpha\dot\alpha} := (\boldsymbol{\gamma}^\mu)_{\alpha\dot\alpha}\,\partial_\mu$ generate translations in $\mathbb{R}^{t,s}$, with $(\boldsymbol{\gamma}^\mu)_{\alpha\dot\alpha}$ the Dirac matrices of the numerical coefficients (Clebsch–Gordan coefficients) that specify the pairing $\mathscr{S}\times\bar{\mathscr{S}} \to \mathbb{R}^{t,s}$ of $\mathrm{Spin}(t,s)$ representations, written in a particular basis for $\mathscr{S}$, $\bar{\mathscr{S}}$ and $\mathbb{R}^{t,s}$. Let $q := \dim(\mathscr{S}) = \dim(\bar{\mathscr{S}})$. The corresponding flat ($\mathbb{R}^{t,s|q,q}$-like) superspace is then parametrized by local coordinates $(x^\mu|\theta^{\alpha\, i}, \bar\theta_i^{\dot\alpha})$, where the θ's and $\bar\theta$'s all mutually anticommute.

2. Generalizing the standard approach for the $N = 2$ cases [12–15,17,18], we introduce the following holomorphic $\mathbb{C}^N$-family of operators (suppressing the $\mathscr{S}, \bar{\mathscr{S}}$-basis indices $\alpha, \dot\alpha$):

$$\mathfrak{D} := \tfrac{1}{\sqrt{N}} z^i D_i, \qquad \mathfrak{D}_{(k)} := \mathfrak{D} - \sqrt{N}\, z^{\underline{k}} D_{\underline{k}}, \tag{2a}$$

$$\bar{\mathfrak{D}} := \tfrac{z^1\cdots z^N}{\sqrt{N} z^i}\, \bar{D}^i, \qquad \bar{\mathfrak{D}}^{(k)} := \bar{\mathfrak{D}} - \sqrt{N}\, \tfrac{z^1\cdots z^N}{z^{\underline{k}}}\, \bar{D}^{\underline{k}}, \tag{2b}$$

where summation is implied over repeated indices (here, i), but not over underlined indices (here, $\underline{k}$). Notice that of the N operators $\mathfrak{D}_{(K)}$ (and likewise for the $\bar{\mathfrak{D}}^{(k)}$'s), only $N-1$ are linearly independent:

$$\sum_{k=1}^{N} \mathfrak{D}_{(k)} = 0 = \sum_{k=1}^{N} \bar{\mathfrak{D}}^{(k)}. \tag{3}$$

Furthermore, in simple supersymmetry ($N = 1$), $\mathfrak{D} = z^1 D_1$, $\bar{\mathfrak{D}} = \bar{D}^1$ while $\mathfrak{D}_{(1)} = 0 = \bar{\mathfrak{D}}^{(1)}$. Finally, over the projective space $\mathbb{P}^N = \mathbb{C}^N/\mathbb{C}^*$ parametrized by $(z^1,\cdots,z^N) \simeq (\lambda z^1,\cdots,\lambda z^N)$ with $\lambda \in \mathbb{C}^*$, $\mathfrak{D}, \mathfrak{D}_{(k)}$ and $\bar{\mathfrak{D}}, \bar{\mathfrak{D}}^{(k)}$ are $\Gamma\big(\mathscr{O}(1)\big)$-linear combinations of the D_i's and $\Gamma\big(\mathscr{O}(N-1)\big)$-linear combinations of the $\bar{D}^i$'s, respectively.

Straightforward calculation produces that:

$$\{\mathfrak{D}, \bar{\mathfrak{D}}\} = 2\mathrm{i}(z^1\cdots z^N)\partial\!\!\!/ \quad \text{and} \quad \{\mathfrak{D}_{(k)}, \bar{\mathfrak{D}}^{(\ell)}\} = (N\delta_k^\ell - 1)\, 2\mathrm{i}(z^1\cdots z^N)\partial\!\!\!/ \tag{4}$$

are the only non-zero anticommutators among the operators (2). In particular, the four subsets of $N+1$ operators:

$$\begin{gathered}\{\mathfrak{D}, \mathfrak{D}_{(1)}, \mathfrak{D}_{(2)}, \ldots, \mathfrak{D}_{(N)}\}, \qquad \{\mathfrak{D}, \bar{\mathfrak{D}}^{(1)}, \bar{\mathfrak{D}}^{(2)}, \ldots, \bar{\mathfrak{D}}^{(N)}\},\\ \{\bar{\mathfrak{D}}, \mathfrak{D}_{(1)}, \mathfrak{D}_{(2)}, \ldots, \mathfrak{D}_{(N)}\}, \qquad \{\bar{\mathfrak{D}}, \bar{\mathfrak{D}}^{(1)}, \bar{\mathfrak{D}}^{(2)}, \ldots, \bar{\mathfrak{D}}^{(N)}\},\end{gathered} \tag{5}$$

are the maximal mutually anticommuting subsets of (2), each with N linearly independent operators owing to the relation (3).

[1] Depending on $1+s$ (mod 8), $\mathscr{S}$ and $\bar{\mathscr{S}}$ may be the same, complex-conjugate or dual to each other [21,22]; we will denote them distinctly for maximal generality.

3. Physically usable Lagrangians may now be constructed from general functions over the extended superspace $(x^\mu|\theta^i,\bar\theta_i|z^1,\dots,z^N)$ and their Laurent expansions in $z^1\cdots z^N$:

$$f(x^\mu|\theta^i,\bar\theta_i|z^1,\dots,z^N)=\sum_{\vec n} f_{\vec n}(x^\mu|\theta^i,\bar\theta_i)\,\mathbf{z}^{\vec n},\qquad \mathbf{z}^{\vec n}:=(z^1)^{n_1}\,(z^2)^{n_2}\cdots(z^N)^{n_N}. \tag{6}$$

Since the supersymmetry generators Q_i, $\bar Q^i$ anticommute with the basis $\{D_i,\ \bar D^i,\ \partial\!\!\!/\}$, all superdifferential constraints using these operators as well as (2) are supersymmetry invariant. In particular, generalizing the $N=2$ constructions of Refs. [12–15,17,18], we consider superfields defined to satisfy the superconstraints

$$N\text{-projective}:\qquad \mathfrak{D}\Theta=0=\bar{\mathfrak{D}}^{(k)}\Theta,\quad k=1,\dots,N. \tag{7}$$

Owing to the linear dependence (3), these restrict the function Θ to depend on only a half of the $2q{\cdot}N$ spinorial coordinates $\theta^i,\bar\theta_i$, and in a z-dependent way. In terms of the Laurent expansion (6), the superdifferential constraints (7) produce $q+q{\cdot}(N-1)$ chain relations:

$$\textstyle\sum_{i=1}^N D_i\,\Theta_{(\vec n-\vec\delta_i)}=0\,, \tag{8}$$

$$\bar D^k\,\Theta_{(\vec n-\vec 1+\vec\delta_k)}=\bar D^{(k-1)}\,\Theta_{(\vec n-\vec 1+\vec\delta_{(k-1)})},\qquad k=2,\cdots,N\,, \tag{9}$$

where $\vec\delta_k=(\delta^1_k,\delta^2_k,\cdots,\delta^N_k)$ and $\vec 1=(1,\cdots,1)$, so $\mathbf{z}^{\vec\delta_i}=z^i$ and $\mathbf{z}^{\vec 1}=z^1\cdots z^N$. For example, when $N=2$, these become (restoring the spinorial indices $\alpha,\dot\alpha$)

$$D_2\,\Theta_{(n_1,n_2-1)}=-D_1\,\Theta_{(n_1-1,n_2)},\qquad \bar D^2\,\Theta_{(n_1-1,n_2)}=\bar D^1\,\Theta_{(n_1,n_2-1)}. \tag{10}$$

In general, the $q{\cdot}N$ constraints (7) restrict Θ to depend on only half of the $2q{\cdot}N$ fermionic coordinates, $\theta^i,\bar\theta_i$, albeit in a somewhat lopsided manner: The chain of Eqs. (9) determines the dependence of Θ on $\bar\theta_2,\cdots,\bar\theta_N$ in terms of its dependence on $\bar\theta^1$. However, the constraint (8) may be solved to express the dependence of Θ only on, say, θ^N in terms of its dependence on $\theta^1,\cdots,\theta^{N-1}$.

By being defined by a linear system of first order superdifferential constraints as well as the uniqueness of the product of Laurent series, the N-projective superfields (7) form a ring: the product of two N-projective superfields is also N-projective. We then further understand any analytic function of N-projective superfields $f(\Theta)$ to be defined in terms of its multi-MacLaurin expansion.

4. Consider then superspace integrals of the form:

$$\mathscr{L}:=\int\prod_{i=1}^{N-1}\mathrm{d}^q\theta^i\,\mathrm{d}^q\bar\theta_1\ f_{\vec n}(\Theta_1,\Theta_2,\dots)=\Big(\prod_{i=1}^{N-1}\wedge^q D_i\Big)\wedge\big(\wedge^q\,\bar D^1\big)\ f_{\vec n}(\Theta_1,\Theta_2,\dots)\big|, \tag{11}$$

where the trailing "|" denotes setting $\theta^{\alpha\,i},\bar\theta_{\dot\alpha\,i}\to 0$ [1,3]. Here, $f(\Theta_1,\Theta_2,\dots)$ is an analytic function of the N-projective superfields $\Theta_1,\Theta_2,\dots$, and $f_{\vec n}(\Theta_1,\Theta_2,\dots)$ is the coefficient of $\mathbf{z}^{\vec n}$ in the multi-Laurent expansion of this function. Integrals over the spinorial coordinates θ^i and $\bar\theta_i$ are the Berezin integrals, for each one of which

$$\int\mathrm{d}\theta^{\alpha\,i}\ f(\theta^{\alpha\,i})=\frac{\partial}{\partial\theta^{\alpha\,i}}\ f(\theta^{\alpha\,i})=D_{\alpha\,i}\ f(\theta^{\alpha\,i})\big|. \tag{12}$$

The q-fold such integration over each copy of $\mathscr{S}$ and $\bar{\mathscr{S}}$ then produces the displayed $q{\cdot}N$-fold superderivative, antisymmetrized owing to the anticommutivity of the D_i's and the $\bar D^i$'s.

A straightforward extension of the explicit argument presented in Ref. [14] then proves that $\mathscr{L}$ is supersymmetric for any particular $\vec{n}$: The superspace (super)derivative representations of the supersymmetry generators satisfy $Q_i - \mathrm{i}D_i = 2\bar{\theta}_i \cdot \bar{\partial}$ and $\bar{Q}^i - \mathrm{i}\bar{D}^i = 2\theta^i \cdot \partial$. Since these differences vanish upon the $\theta^{\alpha\, i}, \bar{\theta}_{\dot{\alpha}\, i} \to 0$ evaluation, the action of the supersymmetry transformation operator $\delta_Q(\epsilon) := -\mathrm{i}(\epsilon^i \cdot Q_i + \bar{\epsilon}_i \cdot \bar{Q}^i)$ on (11) inserts the linear combination $\delta_Q(\epsilon) \simeq (\epsilon^i \cdot D_i + \bar{\epsilon}_i \cdot \bar{D}^i)$ into this superdifferential expression. The half of $\delta_Q(\epsilon) \simeq (\epsilon^i \cdot D_i + \bar{\epsilon}_i \cdot \bar{D}^i)$ that involves superderivatives already explicitly present in the (11) annihilates $\mathscr{L}$ owing to the anticommutivity and so nilpotency of the superderivatives. The other half involves superderivatives the action of which upon $f_{\vec{n}}(\Theta_1, \Theta_2, \ldots)$ the chain relations (8)–(9) convert into the action of the previous half of the superderivatives upon Laurent expansion terms adjacent to $f_{\vec{n}}(\Theta_1, \Theta_2, \ldots)$. These terms then also vanish owing to the anticommutivity and so nilpotency of the superderivatives.

Adding to (11) its hermitian conjugate produces a real functional of the component fields of the superfields Θ_a, which may serve as a supersymmetric Lagrangian.

5. For *simple* ($N = 1$) supersymmetry in 3+1-dimensional spacetime where $q = 2$, the spinorial integration in (11) reduces to $\int \mathrm{d}^2\bar{\theta}$, so that (11) reproduces (the conjugate of) the familiar "superpotential F-terms." The single z-variable in (2) now parametrizes $\mathbb{P}^0$ and so is trivial.

In the simplest *extended* supersymmetry case, $N = 2$, the spinorial integration becomes the complete integration over the "first" spinorial coordinates, $\int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}$, so that (11) reproduces the familiar "D-terms." As discussed in Refs. [12–15,17,18], the z-variables now provide the homogeneous coordinates of a $\mathbb{P}^1 \simeq S^2$, and suitable Lagrangians are found in the form

$$\mathscr{L} = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{\varepsilon_{ij}\, z^i \mathrm{d}z^j}{z^1 z^2} (\wedge^2 D_1) \wedge (\wedge^2 \bar{D}^1)\; f(\Theta_1, \Theta_2, \ldots)\big|, \tag{13}$$

which evidently provides a residue-like localization of the Berezin integrals to the divisor $[z_1 z_2 = 0] \subset \mathbb{P}^1$. That is, the integral (13) is $(N{=}2)$-supersymmetric over $\mathbb{P}^1 \smallsetminus [z_1 z_2 = 0]$ since: #**1**: the integrand is N-projective (7) and #**2**: the integral is $(N{=}1)$-supersymmetric at $[z_1 z_2 = 0]$. The location of the poles could easily be modified by replacing (13) with the integral

$$\mathscr{L}_\phi = \frac{1}{2\pi \mathrm{i}} \oint_{C(\phi)} \frac{\varepsilon_{ij}\, z^i \mathrm{d}z^j}{\phi(z)} (\wedge^2 D_1) \wedge (\wedge^2 \bar{D}^1)\; f(\Theta_1, \Theta_2, \ldots)\big|, \tag{14}$$

where $\phi(z)$ is a quadratic function over $\mathbb{P}^1$, i.e., $\phi(z) \in \Gamma\big(\mathscr{O}(2)\big)$ and the contour $C(\phi)$ encircles the zero-set $\phi^{-1}(0) \in \mathbb{P}^1$. Since $\mathscr{O}(2)$ is the anticanonical bundle of $\mathbb{P}^1$, both $\phi^{-1}(0) \subset \mathbb{P}^1$ and $\mathbb{P}^1 \smallsetminus \phi^{-1}(0)$ are (admittedly extremely simple) Calabi–Yau n-folds: $\phi^{-1}(0) \subset \mathbb{P}^1$ is the compact Calabi–Yau 0-fold, while $\mathbb{P}^1 \smallsetminus \phi^{-1}(0)$ is the non-compact Calabi–Yau 1-fold [23,24].

6. Notice that the degree of $\phi(z)$ in (13) is *not* determined by the overall degree of the superderivatives $(\wedge^2 D_1) \wedge (\wedge^2 \bar{D}^1)$ when re-expressed in terms of the superderivatives (2) complementary to those that annihilate the integrand as per (7); in (13), that degree is 2·2—owing to $q = \dim(\mathscr{S}) = \dim(\bar{\mathscr{S}}) = 2$. Notice that (7) could have equivalently been replaced by the $\star : \mathfrak{D}, \bar{\mathfrak{D}}^{(k)} \leftrightarrow \bar{\mathfrak{D}}, \mathfrak{D}_{(k)}$ conjugate superconstraints. This degree of the superderivative operators in (11) and its $\star$-conjugate analogue would be

$$2q(N-1) \qquad \text{vs.} \qquad q(N^2-2N+2) \tag{15}$$

which equal only for $N=2$. Indeed, the degree of $\phi(z)$ is determined by the degree of the particular term in the multi-Laurent expansion of $f(\Theta_1, \Theta_2, \ldots)$ that the integral isolates, which is for a variety of reasons routinely chosen to be 0 [12–15,17,18]. This then sets the degree of the measure of the contour integral (13) also to be 0.

The foregoing then straightforwardly generalizes to the simple $N>2$ cases where the $2q\cdot N$ superdifferentials are fibered holomorphically over $\mathbb{P}^{N-1}$ as in (2), and N-supersymmetric Lagrangians are constructed as the real parts of

$$\mathscr{L}_\phi = \frac{1}{2\pi\mathrm{i}} \oint\limits_{C(\phi)} \frac{z\mathrm{d}^{N-1}z}{\phi(z)} \Big(\prod_{i=1}^{N-1} \wedge^q D_i\Big) \wedge (\wedge^q \bar{D}^1)\, f(\Theta_1, \Theta_2, \ldots)\big|, \tag{16}$$

where the (multi-)contour integral is of the Atiyah–Bott–Gårding–Candelas type [25,26] and computes the residue at the zero-set $\phi^{-1}(0) \subset \mathbb{P}^{N-1}$; see also Ref. [27]. With $\phi(z)$ a degree-N multinomial the multi-contour integration measure has again degree 0. With the integrand $f(\Theta_1, \Theta_2, \ldots)$ being any analytic function of N-projective superfields (7), the integrals (16) are N-supersymmetric. In fact, this latter condition can be weakened: it suffices for $f(\Theta_1, \Theta_2, \ldots)$ to satisfy the weaker conditions:

$$\mathfrak{D} f(\Theta_1, \Theta_2, \ldots) = \partial\!\!\!/ \cdot \mathcal{K} \qquad \text{and} \qquad \bar{\mathfrak{D}}^{(k)} f(\Theta_1, \Theta_2, \ldots) = \partial\!\!\!/ \cdot \bar{\mathcal{K}}^{(k)}, \quad k = 1, \ldots, N, \tag{17}$$

for some functional expressions $\mathcal{K}, \bar{\mathcal{K}}^{(1)}, \ldots, \bar{\mathcal{K}}^{(N)}$, local in the superfields $\Theta_1, \Theta_2, \ldots$ The chain relations (8)–(9) will then again replace the superderivatives of $f(\Theta_1, \Theta_2, \ldots)$ which do not manifestly appear in (11) with those that do, up to additional terms from (17) that are total spacetime derivatives. General N-extended supersymmetry then transforms the integrals (16) with $f(\Theta_1, \Theta_2, \ldots)$ satisfying (17) into a sum of terms that either vanish owing to duplicate appearance of some of the (anticommuting) superderivatives or are total spacetime derivatives owing to (17). As usual in field theory, the latter terms are assumed to vanish upon spacetime integration, so that the Hamilton action computed from the integrals (16) are invariant under supersymmetry.

In fact, this can be generalized to Calabi–Yau hypersurfaces in Fano $(N-1)$-folds $\mathscr{Y} \subset \mathscr{X}$, provided $\mathscr{X}$ is equipped with a line bundle L with at least N independent sections that can serve as local coordinates on $\mathscr{X}$:

Claim 1. *Let $\mathscr{X}$ be a Fano N-fold, K^* its canonical bundle, and L a line bundle that has N linearly independent sections z^i. Define $\mathfrak{D}, \mathfrak{D}_{(k)}$ and $\bar{\mathfrak{D}}, \bar{\mathfrak{D}}^{(k)}$ for $k = 1, \ldots, N$ as in (2), to be $\Gamma(L)$-linear combinations of the superderivatives D_i's and $\Gamma(K^* L^{-1})$-linear combinations of the conjugate superderivatives $\bar{D}^i$'s, respectively. The integral*

$$\mathscr{L}_{\mathscr{Y}} = \operatorname*{Res}_{\mathscr{Y}\subset\mathscr{X}} \Big[\Big(\prod_{i=1}^{N-1} \wedge^q D_i\Big) \wedge (\wedge^q \bar{D}^1)\, f(\Theta_1, \Theta_2, \ldots)\big|\Big], \tag{18}$$

is N-supersymmetric if the integrand $f(\Theta_1, \Theta_2, \ldots)$ satisfies the conditions (17).

Even upon some additional restriction for physical applications, the computer-aided technology that created about half a billion Calabi–Yau 3-folds [11] and the conjecture that the number of their topological types is in fact infinite[2] [28], would seem to hereby insure a vast variety

[2] If true, Reid's conjecture would certainly imply the same for all Calabi–Yau n-folds with $n > 3$.

of N-supersymmetric Lagrangians, parametrized by non-compact Calabi–Yau $(N-1)$-folds and their Calabi–Yau $(N-2)$-fold completions.

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